Direct Solution of Differential Equations Using a Wavelet-Based Multiresolution Method

Rodrigo Bird Burgos¹, Raul Rosas e Silva¹ and Marco Antonio Cetale Santos²

¹ Pontifícia Universidade Católica do Rio de Janeiro, Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro, Brasil, rburgos@esp.puc-rio.br, raul@puc-rio.br
² LAGEMAR, UFF, Av. Gen. Milton Tavares de Souza, s/n - Niterói, RJ, Brasil, cetale@igeo.uff.br

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Abstract. The use of multiresolution techniques and wavelets has become increasingly popular in the development of numerical schemes for the solution of partial differential equations (PDEs). Therefore, the use of wavelets as basis functions in computational analysis holds some promise due to their compact support, orthogonality, localization and multiresolution properties, especially for problems with local high gradient, which would require a dense mesh in traditional methods, like the FEM. Another possible advantage is the fact that the calculation of the integrals of the inner products of wavelet basis functions and their derivatives can be made by solving a linear system of equations, thus avoiding the problem of approximating the integral by some numerical method. These inner products were defined as connection coefficients and they are employed in the calculation of stiffness, mass and geometry matrices. In this work, the Galerkin Method has been adapted for the direct solution of differential equations in a meshless formulation using interpolating wavelets (Interpolets). This approach enables the use of a multiresolution analysis. One and two-dimensional examples are proposed.

Introduction

The use of wavelet-based numerical schemes has become popular in the last two decades. Wavelets have several properties that are especially useful for representing solutions of partial differential equations (PDEs), such as orthogonality, compact support and exact representation of polynomials of a certain degree. Their capability of representing data at different levels of resolution allows the efficient and stable calculation of functions with high gradients or singularities [1].

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they appear naturally when applying a numerical method for the solution of a PDE. Due to some properties of wavelet functions, these coefficients can be obtained by solving an eigenvalue problem.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of wavelets with interpolating properties, later called Interpolets [2]. Unlike Daubechies’ wavelets [3], Interpolets are symmetric, which is especially interesting in numerical analysis.

The use of wavelets as interpolating functions in numerical schemes holds some promise due to their compact support, localization and multiresolution properties. The approximation of the solution can be improved by increasing either the level resolution or the order of the wavelet used.

Two examples were used for validating the proposed method. In a one-dimensional scheme, a beam with a concentrated load was used to test the method’s ability to capture singularities. In a second example, the method was then applied for a thin plate with excellent results.

Wavelet Theory and Method Formulation

Multiresolution Analysis. Multiresolution analysis using orthogonal, compactly supported wavelets has become increasingly popular in numerical simulation. Wavelets are localized in space, which allows the analysis of local variations of the problem at various levels of resolution. In the following expression, known as the two-scale relation, \( a_k \) are the filter coefficients of the wavelet scale function. In general, there are no analytical expressions for wavelet functions, which can be obtained using iterative procedures.
\[ \varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi(2x) \]  

(1)

**Interpolets.** The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition [4]:

\[ \varphi(k) = \delta_{0,k} \begin{cases} 1, & k = 0 \\ 0, & k \neq 0, \quad k \in \mathbb{Z} \end{cases} \]  

(2)

The filter coefficients for Daubechies-Dubuc Interpolets can be obtained by an autocorrelation of the Daubechies filter coefficients. Interpolets satisfy the same requirements as other wavelets, specially the two-scale relation, which is fundamental for their use as interpolating functions in numerical methods. Fig. 1 shows the Interpolet IN8. Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

Figure 1: Interpolet IN8 scaling function with its full support

**Connection Coefficients.** Assuming that a function \( f \) is approximated by a series of interpolating scale functions, the following may be written:

\[ f(\xi) = \sum_k \alpha_k \varphi_k(\xi) \]  

(3)

The process of solving a differential equation requires the calculation of the inner products of the basis functions and their derivatives. These inner products are defined as connection coefficients:

\[ \Gamma_{ij}^{d_1,d_2} = \int \varphi^{(d_1)}(\xi-i)\varphi^{(d_2)}(\xi-j)d\xi \]  

(4)

The values for the limits of the integral in eq (4) depend on which method is used to impose boundary conditions. In this work, the limits are given by \([0,2^m]\), where \( m \) is the wavelet level of resolution. This method allows the use of Lagrange multipliers to deal with boundary conditions, similarly to what is usually done in a meshless scheme [5]. Connection coefficients at level \( m \) can be obtained through the calculation at level 0 thus avoiding its recalculation while increasing the level of resolution. Wavelet dilation and translation properties allow the calculation of connection coefficients within the interval \([0,1]\) to be summarized by the solution of an eigenvalue problem based only on filter coefficients [6].

\[ \left( P \cdot \frac{1}{2^{d_1+d_2-1}} I \right) \Gamma_{i,j}^{d_1,d_2} = 0, \quad P_{i,j} = a_{i-2}a_{i-2} + a_{i-2+1}a_{i-2+1} \]  

(5)

Since eq (5) leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of the so-called moment equation, derived from the wavelet property of exact polynomial representation [7].
Application to the Bending of a Thin Plate

The bending of a thin plate with thickness \( t \) is modeled by the following DE:

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q(x, y), \quad D = \frac{Et^3}{12(1-\nu^2)}
\]  

(7)

Displacement \( w(x,y) \) is modeled using bi-dimensional wavelets, which are products between one-dimensional wavelets:

\[
w(x,y) = \sum_i \sum_j d_{ij} \phi(x-i) \phi(y-j)
\]  

(8)

As done in a traditional Galerkin approach, eq (8) is substituted in the DE and integrated, leading to a system of equations which contains the wavelets’ connection coefficients [8].

\[
kd = f,
\]

\[
k = D \left[ \Gamma_{00}^{00} \otimes \Gamma_{02}^{22} + \nu \left( \Gamma_{02}^{02} \otimes \Gamma_{02}^{02} + \Gamma_{02}^{02} \otimes \Gamma_{02}^{02} \right) \right] + \Gamma_{00}^{00} \otimes \Gamma_{02}^{22} + 2(1-\nu^2) \Gamma_{02}^{02} \otimes \Gamma_{02}^{02},
\]

(9)

\[
f = \int_0^1 q(x) \Phi^T d\xi
\]

The symbol \( \otimes \) indicates Kronecker product. The system is solved using the stiffness and load matrices provided by eq (9) and imposing essential boundary conditions with Lagrange multipliers:

\[
\begin{bmatrix}
    k & -G^T \\
    -G & 0
\end{bmatrix}
\begin{bmatrix}
    \alpha \\
    \lambda
\end{bmatrix} = \begin{bmatrix}
    f \\
    0
\end{bmatrix}
\]

(10)

In eq (10), \( G \) is a matrix associated with boundary conditions and \( \lambda \) is a vector of Lagrange multipliers. The unknowns in vector \( \alpha \) are the interpolating coefficients of the basis functions instead of nodal displacements.

Examples

Fig. 2 shows a simple example of a beam subjected to a concentrated load at its midpoint. This example was formulated in order to verify the ability of the wavelet method to deal with singularities, since the load generates a discontinuity in the shear force diagram.

This example is easily solved by dividing the beam in two elements and applying the load as a nodal force. In this work, since degrees of freedom don’t have a fixed position, the load is transformed into the wavelet space:

\[
q(\xi) = P\delta \left( \frac{\xi}{2} - \frac{1}{2} \right) \rightarrow \int_0^1 q(\xi) \phi(\xi-i) d\xi = P\phi \left( \frac{1}{2} - i \right)
\]  

(11)
The example was solved using the IN8 Interpolet at different levels of resolution and the results for bending moment and shear force diagrams are shown in Fig. 3. It is clear that higher levels of resolution are necessary in order to capture the singularity that occurs where the load is applied. Nevertheless, results are considerably good, since the solution is obtained in wavelet space and no discretization was performed. The discontinuity in the slope of the bending moment is captured even for a low level of resolution.

Finally, to test the possibility of extending the method to two-dimensional problems, a thin plate was modeled using the equations developed in previous sections. Fig. 4 shows a square plate with all edges clamped subjected to a concentrated load applied at its center. The plate was modeled using the IN6 Interpolet at level 3, leading to a total number of 289 degrees of freedom. The result for the central displacement was \( w = -0.00557 \frac{PL^2}{D} \) which represents an error of 0.5% when compared to the exact solution \( w = -0.00560 \frac{PL^2}{D} \). Results were extremely good, considering that a FE mesh using 32x32 plate elements with 12 degrees of freedom each gives an error of 0.7% in the central displacement.

Fig. 5 shows the results for the bending moments \( M_x, M_y \) and the twisting moment \( M_{xy} \). Displacements and moments distribution were obtained using the wavelet’s second derivatives. The errors in the bending moments \( M_x \) and \( M_y \) at the center point were 4%.
Different types of boundary conditions and loadings were tested for a square plate and the values obtained for central displacement are summarized in Table 1. Results were compared with exact solutions given by [9].

<table>
<thead>
<tr>
<th>Boundary Conditions and Loading Type</th>
<th>Exact</th>
<th>WGM</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamped / Uniform</td>
<td>0.00126qL^4/D</td>
<td>0.00126qL^4/D</td>
<td>0.4 %</td>
</tr>
<tr>
<td>Clamped / Concentrated</td>
<td>0.00560PL^2/D</td>
<td>0.00557PL^2/D</td>
<td>0.5 %</td>
</tr>
<tr>
<td>Simply Supported / Uniform</td>
<td>0.00406qL^4/D</td>
<td>0.00406qL^4/D</td>
<td>0.1 %</td>
</tr>
<tr>
<td>Simply Supported / Concentrated</td>
<td>0.01160PL^2/D</td>
<td>0.01156PL^2/D</td>
<td>0.3 %</td>
</tr>
</tbody>
</table>

Table 1: Results for different types of boundary conditions and loadings

Conclusions

This work presented the formulation and validation of the Wavelet-Galerkin Method using Deslauriers-Dubuc Interpolets. It was also shown that wavelets have the ability of capturing discontinuities without the need to place nodes where they occur.

As in the traditional FEM and other numerical methods, the accuracy of the solution can be improved either by increasing the level of resolution or the wavelet order. Sometimes, lower order wavelets at higher resolutions can give better results than higher order wavelets at lower resolutions.
For two-dimensional problems, results for displacements and bending moments were extremely good, although only regular geometry problems were studied. The extension of the method to irregular geometries is still a challenge. Since the unknowns of the method are interpolation coefficients instead of nodal displacements, it is possible to obtain a smooth representation of bending moments even with a reduced number of degrees of freedom.

References


